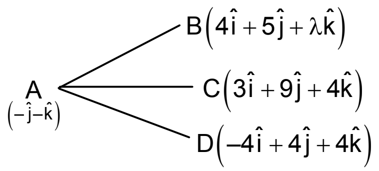
MATHEMATICS

1.



$$\left[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}\right] = \begin{vmatrix} 4 & 6 & \lambda+1 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = 1$$

2. $$\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2 = \begin{vmatrix} 1 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 1 \end{vmatrix} = \frac{3\sqrt{3}-5}{4}$$

Volume = $\frac{1}{6}\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = \frac{\sqrt{3\sqrt{3}-5}}{12}$

3. $$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad k = \frac{9}{2}$$

4. Let $\overrightarrow{PQ} = x\hat{i}+y\hat{j}+z\hat{k}$ $\qquad \therefore \quad d^2 = x^2 + y^2 + z^2$

Now, projection of $\overrightarrow{PQ}$ on xy-plane is d_1 $\qquad \therefore \quad d^2 = d_1^2 + z^2$

similarly $\quad d^2 = d_2^2 + x^2$

$d^2 = d_3^2 + y^2$ $\qquad \therefore \quad d_1^2 + d_2^2 + d_3^2 = 2d^2$

5. $\vec{a}\times\vec{b} = \vec{c}$

$\Rightarrow \quad \vec{a}\times\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\vec{c} \quad \Rightarrow \quad 3\vec{a}-3\vec{b} = -2\hat{i}+\hat{j}+\hat{k} \quad \Rightarrow \quad \vec{b} = \frac{1}{3}\left(5\hat{i}+2\hat{j}+2\hat{k}\right)$

6. $\vec{a}\ \vec{b}\ \vec{c}$ non coplanar

$\Rightarrow \quad \vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}$ are also non-coplanar

$\Rightarrow \quad \vec{a} = \lambda\vec{b}\times\vec{c} + \mu\vec{c}\times\vec{a} + \nu\vec{a}\times\vec{b} \quad \Rightarrow \quad \vec{a}.\vec{a} = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$

similarly μ & ν $\qquad \therefore \quad \vec{a} = \frac{(\vec{a}\ .\ \vec{a})(\vec{b}\times\vec{c})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} + \frac{(\vec{a}\ .\ \vec{b})(\vec{c}\times\vec{a})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]} + \frac{(\vec{a}\ .\ \vec{c})(\vec{a}\times\vec{b})}{[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$

7. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be unit vectors along L_1, L_2, L_3 & L respectively

$\Rightarrow \quad \vec{a}.\vec{d} = \vec{b}.\vec{d} = \vec{c}.\vec{d} \quad \Rightarrow \quad (\vec{a}-\vec{b}).\vec{d} = 0$

$(\vec{b}-\vec{c}).\vec{d} = 0$ & $(\vec{c}-\vec{a}).\vec{d} = 0 \Rightarrow$ is perpendicular to plane π

8. Required area = $\frac{1}{2}\left|\overline{BE}\times\overline{DE}+\overline{EC}\times\overline{DE}\right|$

$= \frac{1}{2}\left|\overline{BC}\times\overline{DE}\right|$

$= \frac{1}{2}\left|\left(-\hat{i}+4\hat{j}\right)\times\left(4\hat{i}-2\hat{j}\right)\right| = 7$

9. $\sin\alpha + 2\sin2\beta + 3\sin3\gamma = 1$ ....(1)

also $|\sin\alpha + 2\sin2\beta + 3\sin3\gamma| \le \sqrt{1+4+9}\ \sqrt{\sin^2\alpha+\sin^2 2\beta+\sin^2 3\gamma}$ as $|\vec{p}\,.\,\vec{q}\,| \le |\vec{p}\,||\,\vec{q}\,|$

$\therefore$ $\sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma \ge \frac{1}{14}$

10. Let $\vec{r}_1 = a\hat{i}+b\hat{j}+c\hat{k}$ & $\vec{r}_2 = x\hat{i}+y\hat{j}+z\hat{k}$

$\because$ $\vec{r}_1.\vec{r}_2 = |\vec{r}_1|\,|\vec{r}_2|$ $\therefore$ $\vec{r}_1 \,||\, \vec{r}_2$ $\Rightarrow$ $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$

11. $\vec{c} = \vec{a}\times\vec{c}+\vec{b}$

$\Rightarrow$ $\left|\vec{c}-\vec{b}\right| = \left|\vec{a}\times\vec{c}\right|$ $\Rightarrow$ $c^2 + 1 - 2\vec{b}.\vec{c} = c^2\sin^2\theta$, where $\theta = \vec{a}\,\hat{}\,\vec{c}$

$\Rightarrow$ $2\vec{b}.\vec{c} = c^2\cos^2\theta + 1$ $\Rightarrow$ $2\vec{b}.\left(\vec{a}\times\vec{c}+\vec{b}\right) = c^2\cos^2\theta + 1$

$\Rightarrow$ $-2\left[\vec{a}\,\vec{b}\,\vec{c}\right] + 2 = c^2\cos^2\theta + 1$ $\Rightarrow$ $2\left[\vec{a}\,\vec{b}\,\vec{c}\right] = 1 - c^2\cos^2\theta \le 1$ $\Rightarrow$ $\left[\vec{a}\,\vec{b}\,\vec{c}\right] \le 1/2$

12. Let $\vec{d} = \alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$

Now $\vec{d}.\vec{b}\times\vec{c} = 2\alpha$

$\vec{d}.\vec{c}\times\vec{a} = 2\beta$

$\vec{d}.\vec{a}\times\vec{b} = 2\gamma$ $\therefore$ $\left[\vec{d}\;\vec{b}\;\vec{c}\right]\vec{a}+\left[\vec{d}\;\vec{c}\;\vec{a}\right]\vec{b}+\left[\vec{d}\;\vec{a}\;\vec{b}\right]\vec{c} = 2\vec{d}$

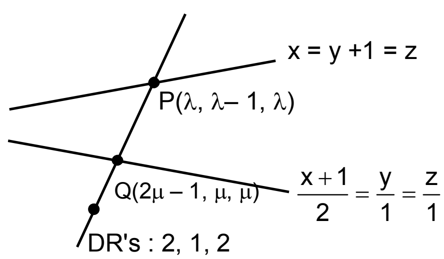
Now, $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)+\left(\vec{b}\times\vec{c}\right)\times\left(\vec{a}\times\vec{d}\right)+\left(\vec{c}\times\vec{a}\right)\times\left(\vec{b}\times\vec{d}\right)$

$= \left[\vec{a}\;\vec{b}\;\vec{d}\right]\vec{c}-2\vec{d}+\left[\vec{b}\;\vec{c}\;\vec{d}\right]\vec{a}-2\vec{d}+\left[\vec{c}\;\vec{a}\;\vec{d}\right]\vec{b}-2\vec{d} = -4\vec{d}$

13. Let equation of plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

Given that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \lambda$ $\because$ fixed point is $\left(\frac{1}{\lambda},\frac{1}{\lambda},\frac{1}{\lambda}\right)$

14.



$\frac{2\mu-\lambda-1}{2} = \frac{\mu-\lambda+1}{1} = \frac{\mu-\lambda}{2}$ $\Rightarrow$ $\lambda = 3$ & $\mu = 1$

15. $\vec{r} = x(\vec{a}\times\vec{b}) + y(\vec{b}\times\vec{c}) + z(\vec{c}\times\vec{a})$

dot with $\vec{a}$, $\vec{b}$ & $\vec{c}$ $\Rightarrow$ $x = \frac{\vec{r}.\vec{c}}{\left[\vec{a}\;\;\vec{b}\;\;\vec{c}\right]}$ and so on

$\Rightarrow \quad \vec{r}\,[\vec{a}\ \vec{b}\ \vec{c}\,] = \frac{1}{2}\,(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\,) \Rightarrow \quad$ Ar $\Delta ABC = |[\vec{a}\ \vec{b}\ \vec{c}\,]\,\vec{r}\,|$

16. Line of intersect of plane (1) and (2) is $\frac{x}{\cos\gamma} = \frac{y}{\cos\beta} = \frac{z}{\cos\alpha}$

which passes through origin and is perpendicular to the normal of the third plane

17. $\cos\alpha = \frac{\vec{r}_1 . \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \frac{(a+c)\cos\theta + b\sqrt{2}\sin\theta + \sqrt{3}(a-c)}{\sqrt{a^2+b^2+c^2}\sqrt{8}}$

for 'α' to be independent of θ, a + c = 0 & b = 0

18. $\vec{r}_1 = \hat{i}\times(\hat{i}+\hat{j}) = \hat{k}$

$\vec{r}_2 = (\hat{i}-\hat{j})\times(\hat{i}+\hat{k}) = -\hat{i}-\hat{j}+\hat{k}$

Now $\vec{a} = \lambda(\vec{r}_1\times\vec{r}_2) = \lambda(\hat{i}-\hat{j})$

19. $\vec{c} = (2\vec{a}\times\vec{b}) - 3\vec{b}$

$\Rightarrow \quad \vec{b}\,.\,\vec{c} = -3|\vec{b}\,|^2$ ...(1)

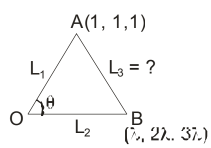
also, $|\vec{c}\,|^2 = 4|\vec{a}\times\vec{b}\,|^2 + 9|\vec{b}\,|^2 - \underbrace{6\vec{b}.(2\vec{a}\times\vec{b})}_{\text{zero}}$ $\Rightarrow \quad |\vec{c}\,|^2 = 4[|\vec{a}\,|^2\,|\vec{b}\,|^2 - (\vec{a}\,.\,\vec{b}\,)^2] + 9|\vec{b}\,|^2$

$\Rightarrow \quad |\vec{c}\,|^2 = 48 \quad \Rightarrow \quad |\vec{c}\,| = 4\sqrt{3}$ ...(2)

$\therefore \quad \cos\theta = \frac{\vec{b}.\vec{c}}{|\vec{b}||\vec{c}|} = \frac{-3|\vec{b}|^2}{|\vec{b}||\vec{c}|} = \frac{-\sqrt{3}}{2}$

20. $\cos\theta = \frac{6}{\sqrt{42}} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{6}}{\sqrt{42}}$

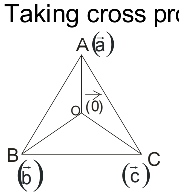
Area $\Delta OAB = \frac{1}{2}$ (OA)(OB) $\sin\theta$



$= \frac{1}{2}\,(\sqrt{3}\,)\,|\lambda|\,(\sqrt{14}\,)\frac{\sqrt{6}}{\sqrt{42}} = \sqrt{6} \quad \Rightarrow \quad \lambda = \pm 2$

21. $\vec{a} + 2\vec{b} + 3\vec{c} = 0$

Taking cross product with $\vec{a}$ and $\vec{b}$,



$\vec{a}\times\vec{b} = \frac{3}{2}(\vec{c}\times\vec{a}) = 3(\vec{b}\times\vec{c})$ Now $\Delta = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| = \frac{1}{2}.2|\vec{a}\times\vec{b}| = \frac{1}{2}.3|\vec{c}\times\vec{a}| = \frac{1}{2}.6|\vec{b}\times\vec{c}|$

22. Let $\hat{r}$ be the new vector $\Rightarrow \quad \hat{r} = \lambda\hat{k} + \mu(\hat{i}+\hat{j})$

$\hat{r}\,.\,\hat{k} = -\frac{1}{\sqrt{2}}$ & $|\hat{r}\,| = 1$ $\quad \lambda = -\frac{1}{\sqrt{2}}$ & $\mu = \pm\frac{1}{2}$

23. Apply VTP to get = $(1 + \hat{a}\,.\,\hat{b}\,)\,(\hat{b} - \hat{a}\,)$

24. Use $\vec{a}.(\vec{b}\times\vec{c}\,) = \vec{b}.(\vec{c}\times\vec{a}\,) = \vec{c}.(\,\vec{a}\times\vec{b}\,)$

25. $(a_1 + a_2) + \sin^2x(a_3 - 2a_2) = 0 \quad \Rightarrow \quad a_1 + a_2 = 0$

& $\quad a_3 - 2a_2 = 0 \quad \Rightarrow \quad \frac{a_1}{-1} = \frac{a_2}{1} = \frac{a_3}{2} = \lambda$

26. $\vec{q} \times \vec{r} = \vec{p}$

$(\vec{q} \times \vec{r}) \times \vec{q} = \vec{p} \times \vec{q} = \vec{r}$

$\Rightarrow \quad |\vec{q}| = 1$ & $\vec{r}\,.\,\vec{q} = 0$ & $\because \vec{q} \times \vec{r} = \vec{p}$

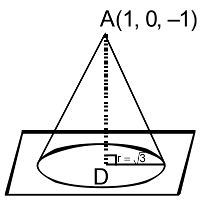
$\Rightarrow \quad |\vec{p}| = |\vec{r}|$

27. The rod sweeps a cone
AD = 1 unit

slant height $\ell = 2$ units $\Rightarrow r = \sqrt{3} \quad \Rightarrow \quad$ volume = $\frac{1}{3}\pi r^2 h = \pi$ cubic units

also, area of circle = $\pi(\sqrt{3})^2 = 3\pi$

& centre is foot of perpendicular of A in plane = $\left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$



28. Volume = $\frac{2\sqrt{2}}{3}$

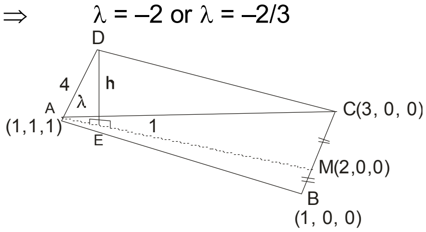
$$\Rightarrow \quad \frac{1}{3}\cdot\frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{matrix}\right\| \times h = \frac{2\sqrt{2}}{3}$$

$\Rightarrow \quad h\,|\hat{j} - \hat{k}| = 2\sqrt{2} \quad \Rightarrow \quad h = 2$

for E, let AE : EM = λ : 1

$\Rightarrow \quad E \equiv \left(\frac{2\lambda+1}{\lambda+1}, \frac{1}{\lambda+1}, \frac{1}{\lambda+1}\right)$ & $(AE)^2 + (ED)^2 = (AD)^2$

$\Rightarrow \quad \lambda = -2$ or $\lambda = -2/3$



29. The plane is perpendicular to the angle bisectors of the line, which are $\frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} \pm \frac{8\hat{i} + \hat{j} - 4\hat{k}}{9}$

30. $\vec{u}\,.\,\hat{i} = |\vec{u}|\cos 60° = \frac{|\vec{u}|}{2} \quad \because \quad$ slope $= \sqrt{3}$

also $|\vec{u} - \hat{i}|^2 = |\vec{u}||\vec{u} - 2\hat{i}| \quad \Rightarrow \quad u^2 + 1 - u = u.\sqrt{u^2 + 4 - 2u} \quad \Rightarrow \quad |\vec{u}| = \sqrt{2} - 1$

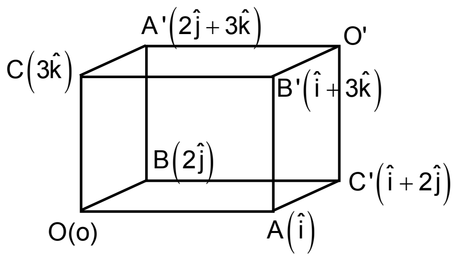
31. $\hat{p} = \dfrac{\hat{a}+\hat{b}}{2\cos\frac{\pi}{6}} = \dfrac{\hat{a}+\hat{b}}{\sqrt{3}}$

Similarly $\hat{q} = \dfrac{\hat{b}+\hat{c}}{\sqrt{3}}$ & $\hat{r} = \dfrac{\hat{c}+\hat{a}}{\sqrt{3}}$

Now $[\hat{p}\ \ \hat{q}\ \ \hat{r}] = \dfrac{1}{3\sqrt{3}}\left[\hat{a}+\hat{b}\quad \hat{b}+\hat{c}\quad \hat{c}+\hat{a}\right] = \dfrac{2}{3\sqrt{3}}\left[\hat{a}\ \ \hat{b}\ \ \hat{c}\right]$

32. Let required vector is $\vec{r} = x\vec{a} + y\vec{b}$

Now $\vec{r}.\hat{c} = \pm\dfrac{1}{\sqrt{3}}$ $\Rightarrow$ $2x - y = \pm 1$

33. 

p.v. of point D = $\overrightarrow{OD} = \dfrac{2\hat{i}+2\hat{j}+3\hat{k}}{2}$

p.v. of point D' = $\overrightarrow{OD} = \dfrac{\hat{i}+2\hat{j}+6\hat{k}}{2}$

now $\cos\theta = \dfrac{\overrightarrow{OD}.\overrightarrow{OD}}{|\overrightarrow{OD}||\overrightarrow{OD}|} = \dfrac{24}{\sqrt{697}}$

34. $\vec{a}\times\left(\vec{a}\times\vec{b}\right) = \left(\vec{a}.\vec{b}\right)\vec{a} - \left(\vec{a}.\vec{a}\right)\vec{b}$

$= -\left(\hat{i}+\hat{j}-\hat{k}\right) - 3\left(\hat{i}-\hat{j}+\hat{k}\right) = -4\hat{i}+2\hat{j}-2\hat{k} = -2\left(2\hat{i}-\hat{j}+\hat{k}\right)$

Required unit vector = $\pm\dfrac{\left(2\hat{i}-\hat{j}+\hat{k}\right)}{\sqrt{6}}$

35. $\left[\vec{a}\times\vec{b}-\vec{c}\times\vec{a}\quad \vec{b}\times\vec{c}+2\vec{a}\times\vec{b}\quad \vec{c}\times\vec{a}-3\vec{b}\times\vec{c}\right]$

$= \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{vmatrix}\left[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\right] = 7\left[\vec{c}\times\vec{a}\quad \vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\right]$

36. $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$ $\Rightarrow$ $c^2 = ab$ $\Rightarrow$ a, c, b are in G.P.

37. $\cos\theta = \dfrac{\vec{p}.\vec{q}}{|\vec{p}||\vec{p}|} = \dfrac{ab+bc+ca}{a^2+b^2+c^2}$

$\because$ we know that $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$ and $(a + b + c)^2 \geq 0$

$\therefore$ $-\dfrac{1}{2} \leq \dfrac{ab+bc+ca}{a^2+b^2+c^2} \leq 1 \Rightarrow -\dfrac{1}{2} \leq \cos\theta \leq 1 \Rightarrow \theta \in \left[0, \dfrac{2\pi}{3}\right]$

38. Normal of plane $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = \hat{i} + 7\hat{j} - 5\hat{k}$

Let equation of plane x + 7y – 5z + d = 0

Now $\left|\frac{1+14-15+d}{\sqrt{75}}\right| = \left|\frac{2+21-5+d}{\sqrt{75}}\right|$

$\Rightarrow$ |d| = |18 + d|

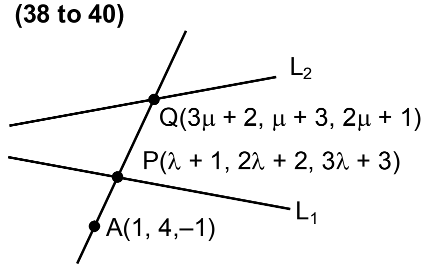
$\Rightarrow$ d = –9

$\therefore$ equation of plane is x + 7y – 5z – 9 = 0.

39. Shortest distance = $\left|\left(-\hat{i}-\hat{j}+2\hat{k}\right).\frac{\left(\hat{i}+7\hat{j}-5\hat{k}\right)}{\sqrt{75}}\right|$

$= \frac{18}{5\sqrt{3}} = \frac{6\sqrt{3}}{5}$

40. **(38 to 40)**



Now $\overrightarrow{AP} \parallel \overrightarrow{AQ}$

$\therefore \quad \frac{\lambda}{3\mu+1} = \frac{2\lambda-2}{\mu-1} = \frac{3\lambda+4}{2\mu+2} \quad \Rightarrow \quad \lambda = 1, -\frac{1}{2}$

but λ = 1 is not possible